Int. J. Solids Structure Vol. 19, No. 7, pp. 601–609, 1983
Printed in Great Britain.

0020-7683/83 $3.00+.00

A NONPARAMETRIC METHOD OF IDENTIFICATION OF VIBRATION DAMPING IN NON-LINEAR DYNAMIC SYSTEMS

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(*Received* 28 *May* 1982; *in revised form* 14 *December* 1982)

Abstract—An original method for identifying non-linear dynamic system damping characteristic is presented. The method suggested allows one to precisely estimate both the magnitude of Coulomb friction and viscoelastic damping which should be of some consequence during investigations of the damping of mechanical vibrations (construction materials). This method has been thoroughly verified on given markedly non-linear analogue systems. Examples of the practical application of the method to mechanics are presented.

1. INTRODUCTION

So far, the measurements of vibration damping in physical dynamic systems have sought to determine individual parameters describing the damping (e.g. logarithmic damping decrement or the so-called "loss factor"[1, 4–6]). Those parameters are closely related to the following three basic damping models: linear viscoelastic damping, viscoelastic damping inversely proportional to frequency, and solid friction. It is obvious that none of the above mentioned three types of models describes exactly damping in a physical dynamic system.

The effect of vibration damping both in construction materials and in other mechanical systems is so complex that its precise mathematical description is very difficult. Different results which are obtained depending on the measuring method used at the onset of the investigation, and the dependence of the parameters obtained by a chosen measuring method on the amplitude and frequency of vibrations, testify to the complex character of the problem.

From the point of view of the theory of identification[7, 8] we are not able to determine optimal values of the constant coefficients for an *a priori* assumed type of model. Known and used in dynamic systems, these identification techniques assume that the type of model is known, and thus their role is only to determine optimal values of model coefficients. Hence, if the model assumed is too simple due to different optimization methods used, considerable discrepancy of results may follow.

This paper presents an unconventional identification method for dynamic systems. Unlike other methods used in dynamic problems, it seeks to determine (at least in part) the form of the mathematical model of the system investigated.

It is assumed that the damping properties of a physical system are sufficiently described by an unknown function $F_t(\dot{x})$ of the velocity $\dot{x}(t)$ of the system. The proposed identification method seeks to precisely determine the form of that function within the arbitrarily adopted (yet finite) range of variation of velocity $\dot{x}$. It is assumed that the dynamic system considered is described by the following differential equation:

$$m\ddot{x} + F_t(\dot{x}) + F_s(x) = p(t) \tag{1}$$

where m ($m > 0$) is a constant parameter denoting mass, $F_t(\dot{x})$ is an unknown function of velocity describing the damping characteristic, and $F_s(x)$ is a function of displacement, describing the elastic characteristic. Although the shape of the functions $F_t(\dot{x})$ and $F_s(x)$ in eqn (1) does not have to be assumed *a priori*, the principal assumption of the method is a parallel structure of the physical model (see Fig. 1).

This means that the present method is not so general as that given in paper [9]. Being somewhat limited, it has, however, two important advantages: (a) it can be applied also when the function $F_t(\dot{x})$ is not continuous, and (b) only a simple experiment is required for the method to be applied.

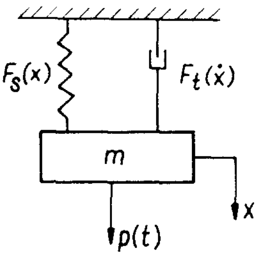

Fig. 1. Diagram of a parallel damped spring-mass system.

It is assumed that within the examined range of velocity $\dot{x}$ variations, this unknown function $F_t(\dot{x})$ has for $\dot{x}=0$ only one point of discontinuity. Functions of such characteristics can accurately be expressed by the following equation:

$$F_t(\dot{x}) = k_0 + K \operatorname{Sgn} \dot{x} + (k \operatorname{Sgn} \dot{x})\dot{x} + \sum_{\nu=1}^{\infty} k_\nu \dot{x}^\nu, \tag{2}$$

where k_0, K, k, k_1, $k_2, \ldots, k_\nu, \ldots$ are constant (Fig. 2).

2. THE METHOD OF IDENTIFICATION

From [2] it follows that we can force harmonic vibrations in non-linear systems (1) by using specially chosen periodic excitations. Let us assume that the function

$$p(t) = 2P_1 \cos \omega t + \Delta p = 2P_1 \cos \omega t + \sum_{i=2}^{\gamma} 2P_i \cos(i\omega t + \varphi_i) \tag{3}$$

is selected in such a way that it forces harmonic vibrations in system (1).

These vibrations are described by the following function:

$$x(t) = x_0 + 2X \cos \psi = x_0 + \mathbf{X}\, e^{j\omega t} + \mathbf{X}^*\, e^{-j\omega t}, \tag{4}$$

where $\psi = \omega t + \alpha$, $\mathbf{X} = X\, e^{j\alpha}$, $\mathbf{X}^* = X\, e^{-j\alpha}$, $j = \sqrt{(-1)}$, x_0-constant. Differential equation (1) for excitation of form (3) and damping function (2) is of the following form:

$$m\ddot{x} + K \operatorname{Sgn} \dot{x} + \sum_{\nu'=1}^{n'} k_{\nu'} \dot{x}^{\nu'} + (k \operatorname{Sgn} \dot{x})\dot{x} + \sum_{\nu''=2}^{n''} k_{\nu''} \dot{x}^{\nu''} + F_s(x) = P_1\, e^{j\omega t} + P_1\, e^{-j\omega t} + \Delta p, \tag{5}$$

where $(\nu' = 1, 3, 5, \ldots, n')$, $(\nu'' = 2, 4, 6, \ldots, n'')$, n' is an arbitrarily large odd number and n'' an arbitrarily large even number.

If the function $x(t)$ is of the form (4), then each of the terms appearing on the l.h.s. of eqn (5) is a periodic function of the variable ψ having the period of 2π. By successively expanding

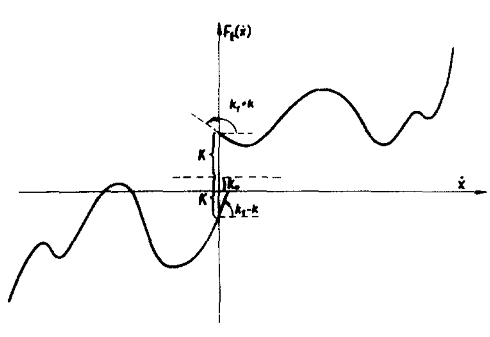

Fig. 2. Example of function $F_t(\dot{x})$.

these components into Fourier series, we get:

$$m\ddot{x}(\psi) = -m\omega^2 X[e^{j\psi} + e^{-j\psi}], \tag{6}$$

$$K \,\mathrm{Sgn}\, \dot{x}(\psi) = \frac{2K}{\pi} j[e^{j\psi} - e^{-j\psi}] + \sum_{i=2}^{\infty} \ldots, \tag{7}$$

$$k_0(\psi) = k_0, \tag{8}$$

$$(k \,\mathrm{Sgn}\, \dot{x})\dot{x}(\psi) = \frac{4X\omega k}{\pi} + 0[e^{j\psi} + e^{-j\psi}] + \sum_{i=2}^{\infty} \ldots, \tag{9}$$

$$F_s[x(\psi)] = f_0(x_0, X) + f_1(x_0, X)\, e^{j\psi} + f_1(x_0, X)\, e^{-j\psi} + \sum_{i=2}^{\infty} \ldots. \tag{10}$$

One can prove (see [3]) that the remaining $n' + n''$ components have the following form:

$$k_{\nu'}\dot{x}^{\nu'} = jk_{\nu'}(\omega X)^{\nu'}\{[\beta_1 e^{j\psi} + \beta_3 e^{3j\psi} + \cdots + \beta_{\nu'-2} e^{(\nu'-2)j\psi} + \beta_{\nu'} e^{\nu' j\psi}] - [\ldots]^*\} \tag{11}$$

$$k_{\nu''}\dot{x}^{\nu''} = k_{\nu''}(\omega X)^{\nu''}\{\beta_0 + [\beta_2 e^{2j\psi} + \cdots + \beta_{\nu''-2} e^{(\nu''-2)j\psi} + \beta_{\nu''} e^{\nu'' j\psi}] + [\ldots]^*\} \tag{12}$$

where the expression $[\ldots]^*$ includes corresponding conjugated terms and $\beta_1, \beta_3, \ldots, \beta_{\nu'}, \beta_0, \beta_2, \ldots, \beta_{\nu''}$ are integers. These numbers are explicitly determined by the corresponding values $\tilde{\beta}_1, \ldots, \tilde{\beta}_{\nu'-2}, \tilde{\beta}_0, \ldots, \tilde{\beta}_{\nu''-2}$ of the components $k_{\nu'-2}\dot{x}^{\nu'-2}$ and $k_{\nu''-2}\dot{x}^{\nu''-2}$ of the lower powers of the velocities $\dot{x}^{\nu'-2}$ and $\dot{x}^{\nu''-2}$ and can be easily calculated from the following formulas

$$\beta_{l'} = -\tilde{\beta}_{l'+2} + 2\tilde{\beta}_{l'} - \tilde{\beta}_{l'-2}, \quad \beta_1 = -\tilde{\beta}_3 + 3\tilde{\beta}_1 \quad (l' = 3, 5, \ldots, \nu') \tag{13}$$

$$\beta_{l''} = -\tilde{\beta}_{l''+2} + 2\tilde{\beta}_{l''} - \tilde{\beta}_{l''-2}, \quad \beta_0 = 2\tilde{\beta}_0 - \tilde{\beta}_2 \quad (l'' = 2, 4, \ldots, \nu'') \tag{14}$$

into which we conventionally introduce supplementary terms $\tilde{\beta}_{\nu'} = \tilde{\beta}_{\nu'+2} = \tilde{\beta}_{\nu'+4} = \cdots = \tilde{\beta}_{\nu''} = \tilde{\beta}_{\nu''+2} = \cdots = 0$ for $l' = \nu', \nu'+2, \ldots$ and $l'' = \nu'', \nu''+2, \ldots$. Formulas (13) and (14) generate two "triangles" of values of the coefficients β. (The first "triangle" (Table 1) is for the odd components $k_{\nu'}\dot{x}^{\nu'}$ and the second one (Table 2) is for the even components $k_{\nu''}\dot{x}^{\nu''}$).

By substituting the terms (6), (7), ..., (12) into the eqn (5) and using the method of harmonic balance, we obtain:

$$k_0 + \frac{4k}{\pi}\omega X + \sum_{\nu''=2}^{n''} \beta_{0\nu''} k_{\nu''}(\omega X)^{\nu''} + f_0(x_0, X) = 0 \tag{15}$$

$$[-m\omega^2 X + f_1(x_0, X)]^2 + \left[\frac{2K}{\pi} + \sum_{\nu'=1}^{n'} \beta_{1\nu'} k_{\nu'}(\omega X)^{\nu'}\right]^2 = P_1^2 \tag{16}$$

Table 1. Initial harmonic component amplitude values for odd powers

Powers \ Harmonic component amplitudes	β_1	β_3	β_5	β_7	β_9	β_{11}	β_{13}	β_{15}	β_{17}	β_{19}	β_{21}
$\dot{x}^1$	1	0	0	0	0	0	0	0	0	0	0
$\dot{x}^3$	3	−1	0	0	0	0	0	0	0	0	0
$\dot{x}^5$	10	−5	1	0	0	0	0	0	0	0	0
$\dot{x}^7$	35	−21	7	−1	0	0	0	0	0	0	0
$\dot{x}^9$	126	−84	36	−9	1	0	0	0	0	0	0
$\dot{x}^{11}$	462	−330	165	−55	11	−1	0	0	0	0	0
$\dot{x}^{13}$	1716	−1287	715	−286	78	−13	1	0	0	0	0
$\dot{x}^{15}$	6435	−5005	3003	−1365	466	−105	15	−1	0	0	0
$\dot{x}^{17}$	24310	−19448	12376	−6188	2380	−680	136	17	1	0	0
$\dot{x}^{19}$	92378	−75582								−1	0
$\dot{x}^{21}$	3 × 92378 + + 75582					2 × 680 + 136 + 2380					

Table 2. Initial harmonic component amplitude values for even powers

Harmonic components amplitude / Powers	β_0	β_2	β_4	β_6	β_8	β_{10}	β_{12}	β_{14}	β_{16}
$\dot{x}^2$	2	-1	0	0	0	0	0	0	0
$\dot{x}^4$	6	-4	1	0	0	0	0	0	0
$\dot{x}^6$	20	-15	6	-1	0	0	0	0	0
$\dot{x}^8$	70	-56	28	-8	1	0	0	0	0
$\dot{x}^{10}$	252	-210	120	-45	10	-1	0	0	0
$\dot{x}^{12}$	924	-792	495	-220	66	-12	1	0	0
$\dot{x}^{14}$	3432	-3003	2002	-1001	364	-91	14	-1	0
$\dot{x}^{16}$	$3432 + 3003 \times 2$					$2 \times 91 + 14 + 364$			

$$tg(-\alpha) = \frac{\frac{2K}{\pi} + \sum_{\nu'=1}^{n'} \beta_{1\nu'} k_{\nu'} (\omega X)^{\nu'}}{-m\omega^2 X + f_1(x_0, X)}. \tag{17}$$

where $\beta_{0\nu'}$ denotes the coefficient β_0 for the power $\dot{x}^{\nu''}$ while $\beta_{1\nu'}$ denotes the coefficient β_1 for the power $\dot{x}^{\nu'}$. Let us denote the phase resonance frequency by ω_r for which:

$$-m\omega_r^2 X_r + f_1(x_{0r}, X_r) = 0. \tag{18}$$

For the frequency $\omega = \omega_r$ the phase shift $\alpha = -\pi/2$ and the expression (16) takes the form:

$$\frac{2K}{\pi} + \sum_{\nu'=1}^{n'} \beta_{1\nu'} k_{\nu'} (\omega X)_r^{\nu'} = P_1. \tag{19}$$

Taking into account that for the state described by function (4), $\omega X = V$ where V denotes the amplitude of velocity, one can write expressions (15) and (19) as follows:

$$-f_0(x_0, X) = k_0 + \frac{4k}{\pi} V + \sum_{\nu''=2}^{n''} \beta_{0\nu''} k_{\nu''} V^{\nu''} \tag{20}$$

and

$$P_1 = \frac{2K}{\pi} + \sum_{\nu'=1}^{n'} \beta_{1\nu'} k_{\nu'} V_r^{\nu'}. \tag{21}$$

One can see that damping characteristic $F_t(\dot{x})$ explicitly determines the relation $P_1(V_r)$ in the system (1). The relation does not depend on the elasticity characteristic $F_s(x)$ of the system. If the function $F_t(\dot{x})$ is not even ($k_0 = k = k_{\nu''} = 0$), then the relation $P(V_r)$ is a transform of the damping characteristic of the object determined by coefficients $\beta_{1\nu'}$. In this case the relation $x_0(X)$ does not depend on the frequency ω of the excitation. See (20).

From the above considerations one can formulate the following procedure for identifying nonlinear dynamic system damping.

(1) Measure the empirical relation $\hat{P}_1(\hat{V}_r)$ and approximate it as exactly as possible with the function of form (21), i.e. with:

$$P_1 = A_0 + \sum_{\nu'=1}^{n'} A_{\nu'} V_r^{\nu'} \quad (\nu' = 1, 3, 5, \ldots, n') \tag{22}$$

(2) Knowing the values of $A_0, A_1, A_3, \ldots, A_{n'}$ calculate the constants $K, k_1, k_3, \ldots, k_{n'}$.

(3) Carry out the measurements of the empirical relation $\hat{x}_0(\hat{X})$ for a few various values of the frequency ω and check if the relation is the function of ω. If the experiment shows that the relation $\hat{x}_0(\hat{X})$ does not depend on ω, then one should consider the characteristic $F_t(\dot{x})$ as

symmetrical (odd function) and assume that

$$k_0 = k = k_2 = k_4 = \cdots = k_{n''} = 0, \tag{23}$$

thus completing the procedure of identification. If however, it does depend on ω, proceed to step 4. To execute this step one must know the elasticity characteristic $F_s(x)$ of the object investigated.

(4) Determine the constant $f_0(x_0, X)$ of the series (10) by assuming that the response of the system is of the form (4).

(5) Measure the empirical relation $\hat{f}_0(\hat{V})$ and approximate it as accurately as possible with the function of form (20), i.e. with:

$$-f_0 = B_0 + B_1 V + \sum_{\nu''=2}^{n''} B_{\nu''} V^{\nu''} \quad (\nu'' = 2, 4, \ldots, n'') \tag{24}$$

(6) Knowing the values of B_0, B_1, B_2, B_4, . . . , $B_{n''}$ calculate the values k_0, k, k_2, k_4, . . . , $k_{n''}$.

3. EXPERIMENTAL VERIFICATION OF THE METHOD. CONCLUSIONS

The method presented in this paper was applied to the identification of damping in some electronic analog systems built according to eqn (1).

Some results of this investigation have already been published elsewhere[3], where a simplified (initial) version of the method is also presented.† The most complex of the systems identified with this method was an electronic system which was designed to work according to the equation

$$m\ddot{x} + A\dot{x} + E(\operatorname{Sgn}\dot{x})\,e^{\dot{x}^2} - D\dot{x}^3 + B + C\sqrt[3]{(\dot{x}^2)} + cx = p(t) \tag{25}$$

(see Fig. 3). As can be seen, a markedly nonlinear electronic system with a nonsymmetrical damping characteristic $F_t(\dot{x})$ has been built in order to examine the functioning of the method presented. This characteristic was identified by applying only harmonic excitation for which the empirical responses $\hat{x}(t)$, $\hat{\dot{x}}(t)$ and $\hat{\ddot{x}}(t)$ have been obtained as nonharmonic. These responses were considered as those of form (4), i.e. the following value

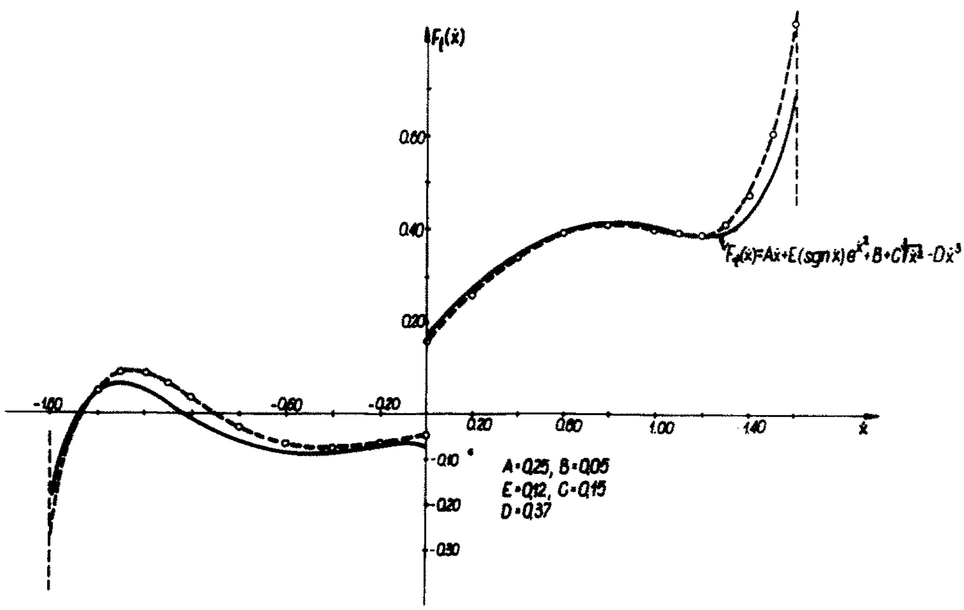

Fig. 3. Illustration of the method functioning on a system with a markedly nonlinear damping characteristic and with Coulomb friction. The solid line (—) is the assumed damping characteristic, and the broken line (---) is the characteristic determined with the identification method.

$$\hat{V} = \frac{1}{4}(\hat{x}_{max} - \hat{x}_{min}) \tag{26}$$

was assumed as the amplitude $\hat{V}$ of velocity $\hat{x}(t)$, where $\hat{x}_{max}$ and $\hat{x}_{min}$ are the measured extreme values of the recorded response $\hat{x}(t)$, while the mean value was calculated with the formula

$$\hat{x}_0 = \frac{1}{2}(\hat{x}_{max} + \hat{x}_{min}). \tag{27}$$

By setting various values of amplitude $\hat{P}_1$ of the harmonic excitation and determining the resonance frequency $\hat{\omega}_r$ for each of them, the relation $\hat{P}_1(\hat{V}_r)$ has been obtained (shown in Fig. 4). This relation has been approximated with function (22) with accuracy up to $n' = 15$ by applying regression analysis (program XDS-2 of ICL, Great Britain). The following approximation of the form (22) has been assumed:

$$A_0 = 0.0662, \quad A_1 = 0.3067, \quad A_3 = -0.8053, \quad A_7 = 1.3816, \quad A_5 = A_9 = A_{11} = A_{13} = A_{15} = 0. \tag{28}$$

Then using the formulas

$$A_0 = \frac{2K}{\pi}, \quad A_{\nu'} = \beta_{1\nu'} k_{\nu'} \quad (\nu' = 1, 3, \ldots, 15) \tag{29}$$

and assuming $\beta_{11} = 1$, $\beta_{13} = 3$, $\beta_{17} = 35$ (see column 1 of Table 1 and the values of (28)), one calculates the following values:

$$K = 0.1039, \; k_1 = 0.3067, \; k_3 = -0.2684, \; k_7 = 0.03947. \tag{30}$$

For the elasticity function $F_s(x) = cx$ of the system (25) the term $f_0(x_0, X)$ takes the form:

$$f_0(x_0, X) = cx_0 \tag{31}$$

(see expressions (10) and (20)). The empirical relation $\hat{f}_0(\hat{V}) = c\hat{x}_0(\hat{V})$ has been obtained by assuming values of amplitude $\hat{P}_1$ and frequency $\hat{\omega}$ so that the values $\hat{V}$ were possibly evenly distributed in the range $2\hat{V} \in (0, 1.6)$. This relation $\hat{x}_0(\hat{V})$ (shown in Fig. 5) has been ap-

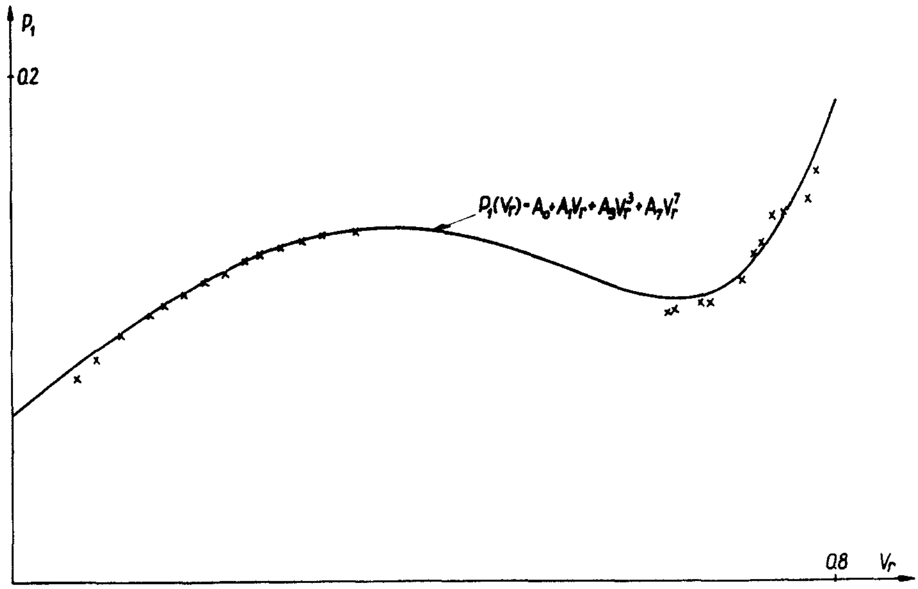

Fig. 4. Empirical relation $\hat{P}_1(\hat{V}_r)$ for a system with a markedly nonlinear damping characteristic and with Coulomb friction. The solid line shows the function $P_1(V_r)$ of the form of (22) approximating the relation.

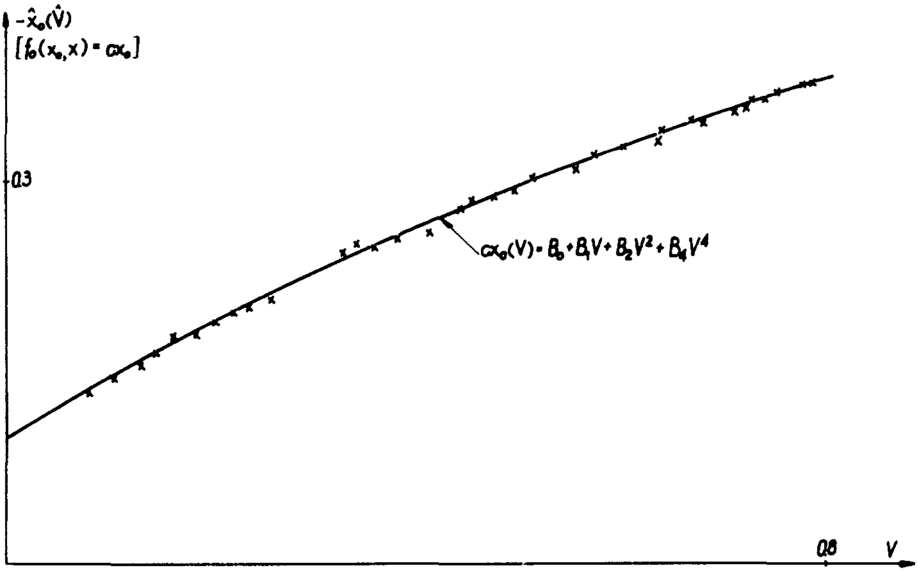

Fig. 5. Empirical relation $\dot{x}_0(\hat{V})$ for a system with a markedly nonlinear damping characteristic and with Coulomb friction. The solid line shows function $-x_0(V)$ of form (24) approximating the relation.

proximated by function (24), accurate up to $n'' = 10$. By applying the method of regression analysis one arrives at the following form of function (24):

$$\begin{aligned} B_0 &= 0.0983\ c \\ B_1 &= 0.503\ c \\ B_2 &= -0.236\ c \\ B_4 &= 0.082\ c \\ B_6 &= B_8 = B_{10} = 0. \end{aligned} \tag{32}$$

The function is shown in Fig. 5 (solid line). Then by comparing expressions (20) and (24) and using the data of (32) one arrives at the following values:

$$\begin{aligned} k_0 &= B_0 = 0.0983 \cdot 0.56 = 0.0550 \\ k &= \frac{\pi B_1}{4} = \frac{3.14}{4} \cdot 0.503 \cdot 0.56 = 0.2211 \\ k_2 &= \frac{B_2}{\beta_{02}} = \frac{1}{2}(-0.236) \cdot 0.56 = -0.0660 \\ k_4 &= \frac{B_4}{\beta_{04}} = \frac{1}{6} \cdot 0.082 \cdot 0.56 = 0.0076, \end{aligned} \tag{33}$$

where the numbers β_{02} and β_{04} are taken from the first column of Table 2. The calculated values (30) and (33) yield the following function of form (2):

$$F_t(\dot{x}) = k_0 + K\ \mathrm{Sgn}\ \dot{x} + (k\ \mathrm{Sgn}\ \dot{x})\dot{x} + k_1\dot{x} + k_2\dot{x}^2 + k_3\dot{x}^3 + k_4\dot{x}^4 + k_7\dot{x}^7, \tag{34}$$

as shown in Fig. 3. Function (34) is an approximation of the damping characteristic of system (25) in the tested range of velocity variation. The accuracy of this approximation depends not only on the application of harmonic excitation is the experiments (for which function (4) does not exactly determine the vibration of the object), but also on the precision with which the analog system is constructed. As one can see (Fig. 3), the presented identification method functions correctly, and even when only harmonic excitation is applied, the gained damping

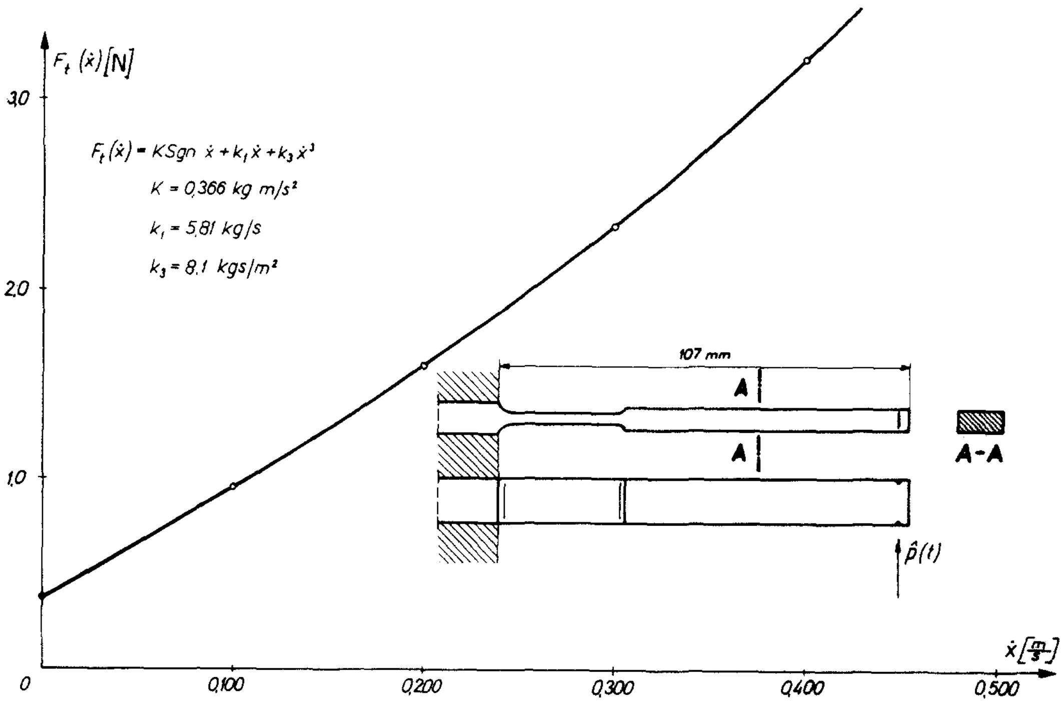

Fig. 6. Damping characteristic of steel beam.

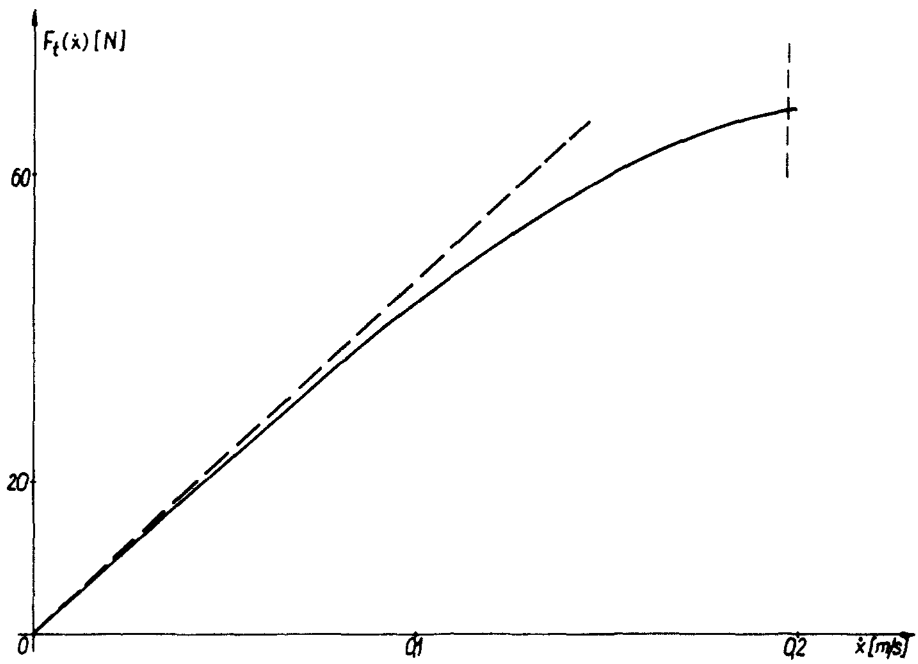

Fig. 7. Damping characteristic of a Trabant-type tire (production of Stomil, 5.20 × 13, pressure 1.3 atm, rotation speed $\omega_0 = 0$).

function approximates accurately enough the real damping characteristic of the system examined. This fact is of great practical importance, as it enables one to apply harmonic excitation generators in experiments, thus making the application of sophisticated periodical excitations unnecessary.

The method presented in this paper has so far been applied to the identification of damping characteristics of cantilever steel beams (Fig. 6) and of a tire (Fig. 7).

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